CLASSES OF BINARY MATROIDS WITH SMALL LISTS OF EXCLUDED INDUCED MINORS

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Abstract. In earlier work, we characterized the class of matroids with no $M(C_4)$ as an induced minor and the class of matroids with no member of $\{M(C_4), M(K_4)\}$ as an induced minor. In this paper, for every two matroids in $\{M(C_4), M(K_4\backslash e), M(K_4), F_7\}$, we determine the class of matroids that have neither of the chosen pair as an induced minor. Additionally, we prove structural lemmas toward characterizing the class of matroids that do not contain $M(K_4)$ as an induced minor.

1. Introduction

The notation and terminology in this paper will follow [13]. Unless stated otherwise, all graphs and matroids considered here are simple. Thus, every contraction of a set from a matroid is immediately followed by the simplification of the resulting matroid. An *induced restriction* of a matroid M is a matroid N that can be obtained from M by restricting to a flat. An *induced minor* of M is a matroid N that can be obtained from M by a sequence of restrictions to flats and contractions, where each such contraction is followed by a simplification. Equivalently, N can be obtained from M by at most one restriction to a flat and at most one contraction followed by a simplification, where these operations can be performed in either order. As noted by Thomas Zaslavsky (private communication), this means that M has N as an induced minor if and only if N is the matroid corresponding to an interval in the lattice of flats of M.

Let M_1 and M_2 be matroids whose ground sets intersect in a set T such that T is a modular flat of M_1, and $M_1|T = M_2|T = N$. The *generalized parallel connection* of M_1 and M_2 across N is the matroid with ground set $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. We denote this matroid by $P_N(M_1, M_2)$ or $P_T(M_1, M_2)$. Note that T may be empty, in which case, $P_T(M_1, M_2) = M_1 \oplus M_2$.

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Given a set $\mathcal{M}$ of binary matroids, we write EXIM($\mathcal{M}$) for the class of binary matroids with no member of $\mathcal{M}$ as an induced minor. In previous work [3, 4], we characterized EXIM($M(C_4)$) and proved the following characterization of EXIM($M(C_4), M(K_4)$).

Theorem 1.1. *The class EXIM($M(C_4), M(K_4)$) consists of the binary matroids that can be obtained from projective geometries over $GF(2)$ by repeated generalized parallel connections across projective geometries over $GF(2)$.*

In Section 2, we consider the other five pairs of rank-3 connected binary matroids and describe the binary matroids in which neither member of the pair occurs as an induced minor. The characterizations of each of these classes include the following.

Theorem 1.2. *The connected members of EXIM($M(K_4\backslash e), M(K_4)$) consist of all projective geometries, all affine geometries of rank at least three, and all circuits with at least three elements.*

Theorem 1.3. *The connected members of EXIM($M(K_4\backslash e), F_7$) consist of F_7^* and all of the matroids $M(C_n)$ and $M(K_n)$ with $n \geq 3$.*

Theorem 1.4. *The connected members of EXIM($M(C_4), M(K_4\backslash e)$) consist of all projective geometries, all projective geometries of rank at least three with a single point deleted, and all cycle matroids of complete graphs.*

Let N be a simple $GF(q)$-represented matroid, that is, a simple $GF(q)$-representable matroid with a given representation. Oxley and Whittle [14] defined the *q-coning*, $A(N)$, of N as the $GF(q)$-representable matroid that is obtained by adding a coloop p to N and then adding every point on each line between p and a point of N. We call p the *tip* (or *apex*) of the cone. This construction was originally introduced by Whittle [20], who called the operation a *q-lift*. Whenever the field is clear, we will write 'coning' in place of 'q-coning'. Similarly, a *tipless coning* of N is the matroid $A(N)\backslash p$. We shall sometimes call $A(N)$ a *tipped coning* of N. For a class $\mathcal{M}$ of $GF(q)$-representable matroids, we write $\widehat{\mathcal{M}}$ for the class of matroids obtained from $\mathcal{M}$ by repeated q-conings. In Section 3, we prove the following result.

Theorem 1.5. *If $\mathcal{M}$ is a class of $GF(q)$-representable matroids closed under taking induced minors, then the class $\widehat{\mathcal{M}}$ is closed under taking induced minors.*

In Section 4, we show that the class of binary matroids that do not contain $M(K_4)$ as an induced minor is closed under taking generalized parallel connections across projective geometries, coning with a tip, and, for a member of the class having no triangles, coning without a tip. Our goal is to eventually prove the following conjecture. A matroid is *triangle-free* if it has no three-element circuits.

Conjecture 1.6. *The class of* 3*-connected binary matroids that do not contain* $M(K_4)$ *as an induced minor is exactly the class of matroids that can be obtained by starting with binary projective geometries and circuits and applying sequences of the following operations:*

(i) generalized parallel connections across projective geometries;
(ii) tipped coning; and
(iii) tipless coning of triangle-free matroids.

In Section 4, we also prove various structural results for matroids that do not contain $M(K_4)$ as an induced minor. The most important of these, which may be of independent interest, is the following.

Theorem 1.7. *Let* M *be a* 3*-connected binary matroid having an element* e *such that neither* M *nor* $M\backslash e$ *has* $M(K_4)$ *as an induced minor. Then every element of* $E(M) - \{e\}$ *is in a triangle with* e*.*

2. Small sets of excluded induced minors

There are exactly four connected rank-3 binary matroids, namely $M(C_4), M(K_4\backslash e), M(K_4)$, and F_7. In this section, we characterize the classes of matroids with pairs of such matroids as excluded induced minors. Unless otherwise stated, the matroids considered in this section will be binary. Recall that, given a set $\mathcal{M}$ of binary matroids, we write EXIM($\mathcal{M}$) for the class of binary matroids with no member of $\mathcal{M}$ as an induced minor. Following Cordovil, Forge, and Klein [1], a matroid M is *chordal* if, for each circuit C with at least four elements, $\mathrm{cl}_M(C) - C \neq \emptyset$. Similarly, an element g of $E(M)$ is a *chord* of a circuit C if $g \in \mathrm{cl}_M(C) - C$. In [4], we proved the following characterization of chordal binary matroids.

Lemma 2.1. *A binary matroid is chordal if and only if it has no* $M(C_4)$ *as an induced minor.*

We use this lemma in the proof of the next result.

Lemma 2.2. *The class EXIM(*$M(C_4), F_7$*) is exactly the class of chordal regular matroids.*

Proof. By Lemma 2.1, EXIM($M(C_4), F_7$) contains all chordal regular matroids. Let M be in EXIM($M(C_4), F_7$). Then M does not have $M(C_4)$ as an induced minor and, by Lemma 2.1, M is chordal. Suppose

M is not regular. Since M is binary, it does not have $U_{2,4}$ as a minor. As M does not have F_7 as an induced minor, it does not have F_7 as a minor. Therefore, since M is not regular, it must have F_7^* as a minor. As F_7^* has $M(C_4)$ as an induced minor, M does not have F_7^* as an induced minor. By the Scum Theorem, M has as a contraction a rank-4 proper extension of F_7^*. Up to isomorphism, $AG(3,2)$ and S_8 are the only simple, rank-4, single-element extensions of F_7^* [17] (see also [13, Lemma 12.2.4]). Both $AG(3,2)$ and S_8 are self-dual and have F_7 as an induced minor. Thus any rank-4 proper extension of F_7^* will have F_7 as an induced minor, a contradiction. □

A graph H is a *parallel extension* of a graph G if $G = H\backslash f$ for an edge f such that f is in a non-trivial parallel class of H. Likewise, a graph H is a *series extension* of a graph G if $G = H/f$ for an edge f of H such that f is in a non-trivial series class of H. A graph G is a *series-parallel network* if it can be obtained from a loop or K_2 by a sequence of operations each of which is a series or parallel extension. A matroid M is called a series-parallel network if M is the cycle matroid of a graph G that is a series-parallel network. Observe that this means that every series-parallel network is a connected matroid.

Lemma 2.3. *The connected members of* $EXIM(M(K_4), F_7)$ *consist precisely of all simple series-parallel networks.*

Proof. Suppose M is a series-parallel network. Then, by a theorem of Dirac [2], M does not have $M(K_4)$ as a minor and therefore M has neither $M(K_4)$ nor F_7 as an induced minor.

Now suppose M is in $\text{EXIM}(M(K_4), F_7)$. Since M is binary, M has no $U_{2,4}$-minor. Moreover, M has no $M(K_4)$-minor. Thus, by, for example, [13, Corollary 12.2.14], M is isomorphic to $M(G)$ for some series-parallel network G. □

Theorem 1.1 and Lemmas 2.2 and 2.3 determine $\text{EXIM}(N_1, N_2)$ for three pairs of matroids from $\{M(C_4), M(K_4), F_7\}$. Each of the remaining pairs $\{N_1, N_2\}$ contains $M(K_4\backslash e)$. The next result will be useful in dealing with these possibilities.

Lemma 2.4. *Let M be a simple connected binary matroid that is not 3-connected and suppose that $|E(M)| \geq 5$. Then*

- *(i) M is a circuit and has $M(C_4)$ as an induced minor; or*
- *(ii) M decomposes as a parallel connection of simple matroids and has $M(K_4\backslash e)$ as an induced minor; or*
- *(iii) M decomposes as a 2-sum of simple matroids and has both $M(C_4)$ and $M(K_4\backslash e)$ as induced minors.*

Proof. If M is a circuit, then M has $M(C_4)$ as an induced minor. Now assume that M is not a circuit. As M is connected but not 3-connected, it has a 2-separation (X, Y). Suppose that $\mathrm{cl}(X)\cap\mathrm{cl}(Y) = \emptyset$. Then, by [4, Lemma 2.3], M has $M(C_4)$ as an induced minor. Moreover, M is the 2-sum of simple matroids M_1 and M_2 that have ground sets $X\cup\{p\}$ and $Y \cup \{p\}$, respectively.

By the assumption that M is not a circuit, at least one of M_1 or M_2, say M_2, is not a circuit. Let C_2 be a minimum-sized circuit of M_2 containing p, and let D_2 be a circuit of M_2 that meets C_2 such that $D_2 - C_2$ is minimal and non-empty. The choice of D_2 implies that $\{C_2-D_2, C_2\cap D_2, D_2-C_2\}$ is a partition of $C_2\cup D_2$ such that the union of every two of these sets is a circuit of M_2. Moreover, each of the sets in $\{C_2-D_2, C_2\cap D_2, D_2-C_2\}$ is a series class of $M_2|(C_2\cup D_2)$ and, since M_2 is simple, at most one of these sets contains a single element. In $M_2|(C_2 \cup D_2)$, we can contract elements to obtain an $M(K_4\backslash e)$-minor having p as an element. Thus, M_2 has an induced minor N_2 using the element p such that N_2 is one of $M(K_4\backslash e)$, $M(K_4)$, or F_7. Let C_1 be a minimum-sized circuit of M_1 containing p and let a_1 and b_1 be elements of $C_1 - p$. Then deleting the elements of M_1 not in C_1 and contracting the elements of $C_1 - \{a_1, b_1, p\}$, we get, as an induced minor N_1 of M_1, a triangle with ground set $\{a_1, b_1, p\}$.

Suppose N_2 is $M(K_4\backslash e)$. Then, by taking the 2-sum of N_1 and N_2, we get an induced minor of M that we can check has $M(K_4\backslash e)$ as an induced minor. Now suppose N_2 is $M(K_4)$ or F_7. Then we can contract an element of N_2 other than p to get a rank-2 matroid N_2' having p in a 2-circuit. The simplification of the 2-sum of N_1 and N_2' with basepoint p is $M(K_4\backslash e)$, so this matroid is an induced minor of M.

We may now assume that $\mathrm{cl}(X) \cap \mathrm{cl}(Y) = \{z\}$ for some element z. For each i in $\{1, 2\}$, let C_i be a circuit in M_i containing z, and let a_i and b_i be distinct elements of C_i-z. Then $M/((C_1\cup C_2)-\{a_1, b_1, a_2, b_2, z\})$ has $\{a_1, b_1, z\}$ and $\{a_2, b_2, z\}$ as triangles and has $\{a_1, b_1, a_2, b_2, z\}$ as a flat. Therefore, M has $M(K_4\backslash e)$ as an induced minor. □

We will often write P_r for the projective geometry $PG(r-1, q)$ when context makes the field clear.

Proof of Theorem 1.2. Let $\mathcal{M}$ be the set of all connected matroids in $\mathrm{EXIM}(M(K_4\backslash e), M(K_4))$. Clearly, all projective geometries, all affine geometries, and all circuits with at least three elements are in $\mathcal{M}$.

Let N be in $\mathcal{M}$. If $r(N) = 3$, then N is either $M(C_4)$ or F_7 and the result holds. Let M be a smallest-rank member of $\mathcal{M}$ such that M is not a projective geometry, an affine geometry, or a circuit. Let $r(M) = r$. Then $r \geq 4$. By Lemma 2.4, we may assume M is 3-connected.

By the choice of M, the matroid M/f is a projective geometry, an affine geometry, or a circuit for each f in $E(M)$. If M/f is a projective geometry for all f in $E(M)$, then, by [3, Lemma 3.2], M is isomorphic to $P_r\backslash P_i$ for some i with $0 \leq i \leq r-1$. If $1 \leq i \leq r-2$, then M has $M(K_4)$ as an induced minor. Therefore, $i = 0$ and M is a projective geometry, or $i = r-1$ and M is an affine geometry. In each case, we obtain a contradiction. Therefore, M has an element f such that M/f is $M(C_n)$ for some $n \geq 4$, or M/f is an affine geometry. In each case, there is at most one triangle containing f otherwise M has $M(K_4\backslash e)$ as an induced minor. Suppose that $M/f \cong M(C_n)$ for some $n \geq 4$. Then $|E(M)| \leq n+2$ and $r(M) = n$. Therefore $r^*(M) \leq 2$, a contradiction since M is 3-connected.

We now know that M/f is an affine geometry for some f in $E(M)$. In this case, as f is in at most one triangle,

$$2^{r-2} + 1 \leq |E(M)| \leq 2^{r-2} + 2. \tag{2.1}$$

Suppose M/g is a projective geometry for some g in $E(M)$. Then $|E(M/g)| = 2^{r-1} - 1$, which contradicts (2.1) since $r \geq 4$. Therefore, for all g in $E(M)$, we must have that M/g is an affine geometry. Assume that M is not an affine matroid and let C be a smallest odd circuit in M. As M is not a circuit, there is an element g in $E(M) - C$. Since M/g is an affine geometry, $g \in \mathrm{cl}_M(C)$, so M has a circuit D such that $g \in D \subseteq C \cup \{g\}$. Then D or $D \triangle C$ is an odd circuit that is smaller than C, a contradiction. We conclude that M is an affine matroid. Thus, if $g \in E(M)$, then M/g has an odd circuit, a contradiction. $\square$

Lemma 2.5. *The 2-connected simple graphs that do not contain $K_4\backslash e$ as an induced minor are cycles and cliques.*

Proof. Suppose G is a 2-connected graph with no $K_4\backslash e$ as an induced minor. By Lemma 2.4, either G is a cycle, or G is 3-connected. Assume that G is not a cycle and not a clique and let x and y be two non-adjacent vertices of G for which the distance d between them is a minimum. Suppose $d \geq 3$. Take a minimum-length (x, y)-path and let x' and y' be two adjacent internal vertices of this path where xx' is an edge. Then the distance from x to y' must be 1 by the choice of x and y. Hence there is a shorter (x, y)-path, a contradiction. Thus $d = 2$. Let w be the internal vertex on a length-2 (x, y)-path. Since G is 3-connected, $G - w$ is 2-connected. Let C be a shortest cycle of G containing x and y and avoiding w. Take Q_1 and Q_2 to be the two (x, y)-paths such that $E(Q_1) \cup E(Q_2) = E(C)$ and $|E(Q_1)| \leq |E(Q_2)|$. If there is an edge from an internal vertex z of Q_1 or of Q_2 to w, then we may contract Q_1 or Q_2 down to two edges, namely xz and zy, and obtain $K_4\backslash e$ as an induced minor of G, a contradiction. Therefore, there are no edges from w to an internal vertex of Q_1 or of Q_2. Then, by contracting Q_1 to a single edge and Q_2 to two edges, we obtain $K_4\backslash e$ as an induced minor of G, a contradiction. □

The next two results, which are due to Hall [5] (see also [13, Proposition 12.2.11]) and Oxley [11, Lemma 3], will be used in the proof of Lemma 2.8.

Theorem 2.6. *If G is a 3-connected graph, then G has no $K_{3,3}$-minor if and only if either G is planar or its associated simple graph is K_5.*

Lemma 2.7. *There is no simple rank-4 regular matroid having $M^*(K_{3,3})$ as a proper restriction.*

Lemma 2.8. *Suppose M is a cographic matroid such that M is not graphic. Then M has $M(K_4\backslash e)$ as an induced minor.*

Proof. Suppose M does not have $M(K_4\backslash e)$ as an induced minor. By Lemma 2.4, we may assume that M is 3-connected. Since M is cographic but not graphic, M is the bond matroid of some non-planar graph G. Then G has $K_{3,3}$ or K_5 as a minor. Suppose G has $K_{3,3}$ as a minor and hence M has $M^*(K_{3,3})$ as a minor. If M does not have $M^*(K_{3,3})$ as an induced minor, then, by Lemma 2.7, M is not regular, a contradiction. Therefore, M has $M^*(K_{3,3})$ as an induced minor, so M has $M(K_4\backslash e)$ as an induced minor, a contradiction. It now follows, by Theorem 2.6, that the simple graph associated with G is K_5. Since $M^*(G) \cong M$, in forming an induced minor of M, any deletion of edges of G is allowed. Therefore, we may assume $G \cong K_5$. However, by contracting a triangle of K_5 and deleting one edge from each of the resulting parallel classes, we obtain the planar dual of $K_4\backslash e$. Hence, by deleting a triad of M and contracting one element from each of the resulting non-trivial series classes, we obtain $M(K_4\backslash e)$ as an induced minor of M, a contradiction. □

We will use the following result of Seymour [16] (see also [13, Corollary 12.2.6]).

Lemma 2.9. *Every binary matroid with no F_7-minor can be obtained from regular matroids and copies of F_7^* by a sequence of direct sums and 2-sums.*

The next result is known as Seymour's Decomposition Theorem [15] (see also [13, Theorem 13.1.1]).

Theorem 2.10. *Every regular matroid M can be constructed by using direct sums, 2-sums, and 3-sums starting with matroids each of which* *is either graphic, cographic, or isomorphic to R_{10} and each of which is a minor of M.*

We now prove the second main theorem of this section.

Proof of Theorem 1.3. Certainly all circuits, all cycle matroids of complete graphs, and F_7^* are contained in the set of connected members of $\text{EXIM}(M(K_4\backslash e), F_7)$.

Now suppose M is a connected member of $\text{EXIM}(M(K_4\backslash e), F_7)$. Assume M is not a circuit, the cycle matroid of a complete graph, or F_7^*. By Lemma 2.4, we may assume M is 3-connected. Since M does not have F_7 as an induced minor, M does not have F_7 as a minor and therefore, by Lemma 2.9, M can be built from regular matroids and copies of F_7^* by 2-sums. Since M is 3-connected and, by assumption, $M \ncong F_7^*$, we deduce M is a regular matroid. Then, by Theorem 2.10, M can be obtained from graphic matroids, cographic matroids, and copies of R_{10} by 2-sums or 3-sums, and M has each starting matroid as a minor. By the construction of 2-sums and 3-sums [13, Theorem 8.3.1 and Proposition 9.3.5], such a minor can be achieved by contracting and simplifying only, M has each starting matroid as an induced minor. If any of the matroids used to obtain M is isomorphic to R_{10}, then M has $M^*(K_{3,3})$ as an induced minor since $R_{10}/x \cong M^*(K_{3,3})$ for all elements x. Therefore, M has $M(K_4\backslash e)$ as an induced minor, a contradiction. If any of the matroids used to obtain M is cographic but not graphic, then, by Lemma 2.8, M has $M(K_4\backslash e)$ as an induced minor, a contradiction. Thus, all of the matroids used to obtain M must be graphic. Any 2-sum or 3-sum of graphic matroids is a graphic matroid. By Lemma 2.5, M is the cycle matroid of either a cycle or a clique, a contradiction. $\square$

For a matroid M and a positive integer k, let (X, G, Y) be a partition of $E(M)$ with $G = \text{cl}(X) \cap \text{cl}(Y)$ such that both $(X \cup G, Y)$ and $(X, Y \cup G)$ are vertical k-separations of M. When this occurs, it is convenient to call (X, G, Y) a vertical k-separation of M.

Recall that a matroid is *round* if, for all positive integers k, it has no vertical k-separations or, equivalently, it has no two disjoint cocircuits.

Lemma 2.11. *Each member of EXIM($M(C_4), M(K_4\backslash e)$) is either disconnected or round.*

Proof. Suppose M is a connected member of $\text{EXIM}(M(C_4), M(K_4\backslash e))$, and that M has a vertical k-separation (X, G, Y) for some $k \geq 2$. First suppose $G = \emptyset$. Then, by [4, Lemma 2.3], M has $M(C_4)$ as an induced minor, a contradiction.

If $r(G) < k-1$, then M/G has a vertical k'-separation (X', G', Y') with $G' = \emptyset$ and, by [4, Lemma 2.3], M/G has $M(C_4)$ as an induced minor, a contradiction. We deduce that $r(G) = k-1$. Let g be an element of G. Then $M' = M/(G - \{g\})$ is a parallel connection of $(M|\mathrm{cl}(X))/(G-g)$, and $(M|\mathrm{cl}(Y))/(G-g)$. By Lemma 2.4, M has $M(K_4\backslash e)$ as an induced minor, a contradiction. □

A flat F in a matroid M is a *connected flat* if $M|F$ is connected.

Corollary 2.12. *If F is a connected flat in a matroid M in the class EXIM$(M(C_4), M(K_4\backslash e))$, then $M|F$ is round.*

McNulty and Wu [7] proved the following result.

Lemma 2.13. *Let M be a 3-connected binary matroid with at least four elements. Then, whenever f and g are distinct elements of M, there is a connected hyperplane of M containing f and avoiding g.*

We now prove the third main result of this section.

Proof of Theorem 1.4. Clearly each of the matroids listed is a connected member of EXIM$(M(C_4), M(K_4\backslash e))$. We now show that these are the only connected matroids in EXIM$(M(C_4), M(K_4\backslash e))$. Let M be a minimum-rank connected member of EXIM$(M(C_4), M(K_4\backslash e))$ that is not one of the listed matroids. Then $r(M) \geq 4$. Assume $r(M) = 4$. By Lemma 2.4, we may assume that M is 3-connected. If M is graphic, then $M \cong M(\mathcal{W}_4)$ or $M \cong M(K_5\backslash e)$, so M has $M(K_4\backslash e)$ as an induced minor, a contradiction. Thus, M is not graphic, so, by a theorem of Tutte [19], either M is not regular, or M is regular and has $M^*(K_{3,3})$ as a minor. In the second case, by Lemma 2.7, $M \cong M^*(K_{3,3})$, so M has $M(K_4\backslash e)$ as an induced minor, a contradiction. We deduce that M is not regular. Then, by another theorem of Tutte [18], M has F_7 or F_7^* as a minor. By a result of Seymour [17] (see also [13, Lemma 12.2.4]), M must have F_7^* as a spanning restriction. As F_7^* has $M(C_4)$ as an induced restriction, $|E(M)| \geq 8$. Thus, as M is simple and $r(M) = 4$, by [13, Lemma 12.2.4] again, M has $AG(3,2)$ or S_8 as a restriction. Since the last two matroids have $M(C_4)$ and $M(K_4\backslash e)$ as induced restrictions, we deduce that $|E(M)| \geq 9$. Thus, M is a proper extension of $AG(3,2)$ or S_8 and hence the complement M^c of M in P_4 is a proper restriction of F_7 or of $M(K_4) \oplus U_{1,1}$, respectively. Since M is not P_4 or a single-element deletion of P_4, we see that $|E(M)| \leq 13$. Now M^c has $U_{2,2}$ or $U_{2,3}$ as a rank-2 flat F. Each of the three projective planes that contain F must contain a point of M^c that is not in F, otherwise M has $M(K_4\backslash e)$ or $M(C_4)$ as an induced restriction. But each of the possibilities for M^c has rank 4 so none is a restriction of F_7. Thus, M^c

is a restriction of $M(K_4) \oplus U_{1,1}$ having at least five points including the point corresponding to $U_{1,1}$. Hence, M^c is $M(K_4\backslash e) \oplus U_{1,1}$, $U_{2,3} \oplus U_{2,2}$, or $M(C_4) \oplus U_{1,1}$. In each case, M has $M(C_4)$ or $M(K_4\backslash e)$ as an induced restriction. We conclude that $r(M) \neq 4$.

We may now assume that $r(M) \geq 5$. By Lemma 2.13, M has a connected hyperplane, H. By the choice of M, it follows that $M|H$ is a projective geometry, a projective geometry with a point deleted, or the cycle matroid of a complete graph. Suppose first that $M|H$ is a projective geometry with one point deleted. Let p be the projective point missing from H and let x be in $E(M) - H$. Suppose the point y on the line from x to p is in $E(M)$. Then, for each element h in H, by the choice of M, the plane spanned by $\{x, h, y\}$ is isomorphic to $M(K_4)$. In particular, for each h in H, the third point on the line spanned by $\{x, h\}$ is in $E(M)$. Hence, $M|\mathrm{cl}_M(H \cup \{x\})$, which equals M, is a projective geometry with one point deleted, a contradiction. We deduce that there is no projective line through p that contains two points of $E(M) - H$. In particular, $y \notin E(M)$.

Let z be in $E(M) - (H \cup \{x\})$. Then there is a point g of H such that $\{x, z, g\}$ is a triangle. However, g is in a rank-4 flat F of H that spans p and $M|F$ is isomorphic to $P_4 - \{p\}$. This implies that $\mathrm{cl}(F \cup x)$ is a connected flat of M. Suppose first that $r(M) > 5$. Then, by the choice of M, the matroid $M|\mathrm{cl}(F \cup \{x\})$ is a projective geometry, a projective geometry with one point deleted, or the cycle matroid of a complete graph. Since $M|\mathrm{cl}(F)$ has F_7 as a minor, $M|\mathrm{cl}(F \cup \{x\})$ cannot be the cycle matroid of a complete graph. Moreover, since $p \notin E(M)$, the matroid $M|\mathrm{cl}(F \cup x)$ is a projective geometry with exactly one point deleted, a contradiction as the point on the line between x and p was assumed to not be in $E(M)$. We deduce that $r(M) = 5$, so $r(H) = 4$. Thus $|H| = 14$. Since no projective line through p has both elements in $E(M) - H$, it follows that $|E(M) - H| \leq \frac{16}{2} = 8$. Hence $|E(M)| \leq 22$.

As $M|H = P_4 - \{p\}$, there is a subset W of H containing g such that $M|W \cong F_7$. Then $\mathrm{cl}(W \cup \{x\})$ is a connected hyperplane H' of M. Evidently, $M|H'$ is isomorphic to P_4 or P_4 with a point deleted. Since $H \cap H' \supseteq W$, equality must hold so

$$\begin{aligned} 22 \geq |E(M)| &\geq |H| + |H'| - |W| \\ &\geq \ 14 \ + \ 14 \ - \ 7 \ = 21. \end{aligned}$$

Hence either $E(M) = H \cup H'$, or M has a unique element u that is not in $H \cup H'$. In each case, consider an element w of W. Then there are elements h and h' of $H - H'$ and $H' - H$, respectively, such that $M|\mathrm{cl}(\{h, h', w\}) \cong M(K_4\backslash e)$ unless $\mathrm{cl}(\{h, h', w\})$ contains u, that is, unless $\mathrm{cl}(\{h, h', w\})$ is not contained in $H \cup H'$. In the exceptional case, we can take an element h_1 of $(H - H') - \mathrm{cl}(\{h, h', w\})$ such that $M|\mathrm{cl}(\{h_1, h', w\}) \cong M(K_4 \backslash e)$ since $\mathrm{cl}(\{h_1, h', w\})$ cannot also contain u. We conclude that $r(M) \neq 5$. It follows that no connected hyperplane of M is a projective geometry with a point deleted.

Next suppose that $M|H$ is a projective geometry. Let x and y be distinct elements of $E(M) - H$. Then there is an element h in H such that $\{x, y, h\}$ is a triangle of M. Since, by Lemma 2.11, M is round, there must be an element z in $E(M) - (H \cup \{x, y\})$. This implies there are elements f_z and g_z in H such that $\{x, z, f_z\}$ and $\{y, z, g_z\}$ are triangles of M. Then $M|\mathrm{cl}(\{x, y, z, f, g_z, h_z\})$ is either $M(K_4)$ or F_7. Let X be a largest subset of $E(M) - H$ containing $\{x, y, z\}$ such that $M|\mathrm{cl}(X)$ is a projective geometry with at most one point deleted. If $X = E(M) - H$, then, since M is round, the cocircuit X is spanning. Thus, M is a projective geometry with at most one point deleted, a contradiction. Let w be in $E(M) - (H \cup \mathrm{cl}(X))$. Consider $M|\mathrm{cl}(X \cup w)$. For any point t of X, there is a point h_t in $H - \mathrm{cl}(X)$ such that $\{t, w, h_t\}$ is a triangle of M. Thus, $\mathrm{cl}(X \cup w)$ is a connected flat of M. As $r(\mathrm{cl}(X \cup w) \cap H) \geq 3$, we see that $M|\mathrm{cl}(X \cup w)$ has F_7 as a restriction. Hence either $\mathrm{cl}(X \cup w) = E(M)$, or $M|\mathrm{cl}(X \cup w)$ is a projective geometry or a projective geometry with a point deleted. The second possibility contradicts the choice of X. Since $\mathrm{cl}(X \cup w) = E(M)$, we see that $\mathrm{cl}(X)$ is a connected hyperplane of M. As $r(\mathrm{cl}(X) \cap H) \geq 3$, we deduce that $M|(\mathrm{cl}(X) \cap H)$ has F_7 as a restriction, so $M|\mathrm{cl}(X)$ is a projective geometry of rank $r(M) - 1$.

We show next that M is a projective geometry by showing that each line through w and a point h of H contains three points of M. This is certainly true if $h \in H - \mathrm{cl}(X)$ because $\mathrm{cl}(X)$ is a projective hyperplane. Now take h in $\mathrm{cl}(X) \cap H$. Let x_1 be a point of $X - H$. Then the third point x_2 on the projective line spanned by $\{x_1, h\}$ is in $E(M)$. Extend $\{x_1, x_2\}$ to a basis B_X of X. Take b in $B_X - \{x_1, x_2\}$. Then $\mathrm{cl}((B_X - b) \cup w)$ is a connected hyperplane of M containing h and w. Since the restriction to this hyperplane has an F_7-restriction, it must be a projective geometry. Thus, the third point on the projective line spanned by $\{w, h\}$ is in $E(M)$. We conclude that M is indeed a projective geometry, a contradiction. We deduce that no connected hyperplane of M is a projective geometry.

We now know that, for every connected hyperplane H of M, the matroid $M|H$ must be isomorphic to the cycle matroid of a complete graph. Hence $|H| = \binom{r(M)}{2}$.

2.14.1. *For all x in $E(M)$, the matroid M/x is the cycle matroid of a complete graph.*

Let x be in $E(M)$. Since M is 3-connected, by Lemma 2.13, M has a connected hyperplane H avoiding x. Suppose that M/x is a projective geometry or a projective geometry with exactly one point deleted. As $M|H \cong M(K_n)$ for some $n \geq 5$, we see that $M|H$ has a flat F such that $M|F \cong M(K_5)$. Viewing M as a restriction of $P_{r(M)}$, there are elements a and b of $\mathrm{cl}_{P_{r(M)}}(F) - F$ such that $\{x, a, a_x\}$ and $\{x, b, b_x\}$ are triangles of $P_{r(M)}$ for some elements a_x and b_x in $E(M) - (H \cup \{x\})$ because M/x is a projective geometry possibly with a point deleted.

Since $M|F \cong M(K_5)$, the set $\mathrm{cl}_{P_{r(M)}}(F) - F$ is a 5-circuit so it does not contain any triangles. Thus, there is a point y in F such that $\{a, b, y\}$ is a triangle of $P_{r(M)}$. If the third point on the line $\mathrm{cl}_{P_{r(M)}}(\{x, y\})$ is in $E(M)$, then $\mathrm{cl}_M(\{x, a_x, b_x\})$ is a flat isomorphic to $M(K_4\backslash e)$, a contradiction. Thus, the third point on $\mathrm{cl}_{P_{r(M)}}(\{x, y\})$ is not in $E(M)$. Let d be a point of $\mathrm{cl}_{P_{r(M)}}(F) - (F \cup \{a, b\})$ such that $\{x, d, d_x\}$ is a triangle in $P_{r(M)}$ for some d_x in $E(M) - H$. Note that such an element must exist since M/x is a projective geometry with at most one point deleted and $|\mathrm{cl}_{P_{r(M)}}(F) - (F \cup \{a, b\})| = 3$.

As the complement of $M(K_5)$ in P_4 is a 5-circuit, the third point t on the line $\mathrm{cl}_{P_{r(M)}}(\{d, y\})$ must be in F, otherwise $\{a, b, d, t\}$ forms a 4-circuit in $\mathrm{cl}_{P_{r(M)}}(F) - F$. Thus, $\mathrm{cl}_M(\{x, d_x, y\})$ is a connected rank-3 flat containing four or five elements, a contradiction. We conclude that M/x is the cycle matroid of a complete graph for all x in $E(M) - H$. Therefore, 2.14.1 holds.

By 2.14.1, every single-element contraction of M is regular. Since M is binary of rank at least five, the Scum Theorem implies that M has neither F_7 nor F_7^* as a minor. Thus, by a theorem of Tutte [18], M is regular. Let H be a connected hyperplane of M, let $C^* = E(M) - H$, and suppose C^* is dependent. Since M is binary, every circuit contained in C^* has even cardinality. Let $\{x_1, x_2, \ldots, x_s\}$ be a circuit C contained in C^*. Because each of $M|H$ and M/x_s is the cycle matroid of a complete graph, the third point y_i on the projective line spanned by $\{x_i, x_s\}$ is in M otherwise M/x_s has more points than $M|H$. Then $M|(C \cup \{y_1, y_2, \ldots, y_{s-1}\})$ is isomorphic to a binary spike of rank $s - 1$ with a tip. As the last matroid has F_7 as a minor, we have a contradiction. We conclude that C^* is independent, and, by Lemma 2.11, M is round, so C^* is spanning. Therefore, $|C^*| = r(M)$. Hence, $|E(M)| = r(M) + \binom{r(M)}{2} = \binom{r(M)+1}{2}$. By [9] (see also [13, Proposition 14.10.3]), as M is a regular matroid with $\binom{r(M)+1}{2}$ elements, $M \cong M(K_{r(M)+1})$, a contradiction. $\square$

3. Conings

In the introduction, we defined the q-coning of a simple $GF(q)$-represented matroid. Suppose now that we begin with a simple $GF(q)$-representable matroid N without a given representation. A *q-coning* of N is any matroid that can be obtained as the q-coning of a particular $GF(q)$-representation of N. Kung [6, p.102] noted that q-conings of two equivalent $GF(q)$-representations of a matroid are isomorphic. In particular, all 2-conings of a simple binary matroid are isomorphic. However, Oxley and Whittle [14] gave an example, for all $q \geq 4$, of non-isomorphic q-conings of a rank-3 simple $GF(q)$-representable matroid.

In the next lemma [20, Lemma 2.3], we summarize some basic properties of q-conings that will be used throughout this section.

Lemma 3.1. *Let N be a simple $GF(q)$-representable matroid, let E be a subset of $PG(r(N),q)$ such that $PG(r(N),q)|E = N_0 \cong N$, and let p be an element of $E(PG(r(N),q))$ that is not in $\mathrm{cl}_{PG(r(N),q)}(E)$.*

(i) If H is a hyperplane of $PG(r(N),q)$ disjoint from p, then $H \cap E(A(N_0))$ is a hyperplane of $A(N_0)$.
(ii) If H' is a hyperplane of $A(N_0)$ disjoint from p, then $A(N_0)|H' \cong N_0 \cong N$.
(iii) If H' is a hyperplane of $A(N_0)$ disjoint from p, then $A(N_0)$ is the q-coning of $A(N_0)|H'$ with tip p.

Next we build towards showing that, for an induced-minor-closed class $\mathcal{M}$ of $GF(q)$-representable matroids, the class $\widehat{\mathcal{M}}$ of matroids that contains $\mathcal{M}$ together with all matroids that can be built from members of $\mathcal{M}$ by repeatedly taking q-conings is also an induced-minor-closed class. Unless specified otherwise, all matroids considered in this section are $GF(q)$-representable.

Lemma 3.2. *If $\mathcal{M}$ is a class of $GF(q)$-representable matroids closed under taking induced minors, then $\widehat{\mathcal{M}}$ is closed under taking contractions.*

Proof. Let M be a smallest-rank member of $\widehat{\mathcal{M}}$ for which there is an element e such that $M/e \notin \widehat{\mathcal{M}}$. As $\mathcal{M}$ is closed under taking induced minors, $M \notin \mathcal{M}$. Thus, $M = A(N)$ for some N in $\widehat{\mathcal{M}}$. If $e = p$, the tip of the coning, then $M/e \cong N$, so $M/e \in \widehat{\mathcal{M}}$, a contradiction. Hence $e \in E(M) - p$. By Lemma 3.1(i), there is a hyperplane H of

M that contains e such that $M|H \cong N$. Let $N' = M|H$. Then N' is in $\widehat{\mathcal{M}}$ and, therefore, N'/e is in $\widehat{\mathcal{M}}$. Thus, by Lemma 3.1(iii), we have $M/e \cong A(N'/e)$, so $M/e \in \widehat{\mathcal{M}}$, a contradiction. $\square$

Lemma 3.3. *If $\mathcal{M}$ is a class of $GF(q)$-representable matroids closed under taking induced restrictions, then $\widehat{\mathcal{M}}$ is closed under taking induced restrictions.*

Proof. Let M be a smallest-rank member of $\widehat{\mathcal{M}}$ having a flat F such that $M|F \notin \widehat{\mathcal{M}}$. Then $M \notin \mathcal{M}$, so $M = A(N)$ for some N in $\mathcal{M}$. Suppose F contains the tip p. Then $F \cap E(N)$ is a flat of N and, as $N \in \mathcal{M}$ and $r(N) < r(M)$, it follows that $N|(F \cap E(N)) \in \mathcal{M}$. Therefore, as $M|F = A(N|(F \cap E(N)))$, we see that $M|F$ is in $\widehat{\mathcal{M}}$, a contradiction. Thus, $p \notin F$. Hence, by Lemma 3.1(ii), M has a hyperplane H avoiding p such that $M|H \cong N$ and $F \subseteq H$. Let $N' = M|H$. Then $N' \in \widehat{\mathcal{M}}$. As $M|F = N'|F$, we see, since $r(N') < r(M)$, that $M|F \in \widehat{\mathcal{M}}$, a contradiction. $\square$

Proof of Theorem 1.5. This is an immediate consequence of combining Lemmas 3.2 and 3.3. $\square$

Given a class $\mathcal{M}$ of $GF(q)$-representable matroids, let $\widecheck{\mathcal{M}}$ be the class of matroids obtained by starting with matroids in $\mathcal{M}$ and applying a sequence of tipless q-conings; let $\widetilde{\mathcal{M}}$ be the class of matroids obtained by starting with matroids in $\mathcal{M}$ and applying a sequence of operations each of which is a tipped q-coning or a tipless q-coning. As an example, we have the following result whose straightforward proof is omitted.

Proposition 3.4. *The class of $GF(q)$-representable matroids that can be obtained by starting with $U_{1,1}$ and applying a sequence of tipless q-conings consists of $U_{1,1}$ and the class of non-empty affine geometries over $GF(q)$.*

If N is an affine geometry over $GF(q)$ with $r(N) \geq 3$ and $e \in E(N)$, then N/e is a projective geometry over $GF(q)$ of rank at least two. Hence, for a class $\mathcal{M}$ of $GF(q)$-representable matroids, the class $\widecheck{\mathcal{M}}$ need not be closed under taking contractions.

The following results are essentially consequences of Lemma 3.1. We omit their proofs because they are so similar to the proofs of Lemmas 3.3 and 3.2.

Lemma 3.5. *If $\mathcal{M}$ is a class of $GF(q)$-representable matroids closed under taking induced restrictions, then $\widecheck{\mathcal{M}}$ is closed under taking induced restrictions.*

Theorem 3.6. *If $\mathcal{M}$ is a class of $GF(q)$-representable matroids closed under taking induced minors, then $\widetilde{\mathcal{M}}$ is closed under taking induced minors.*

Recall that we are writing P_r as an abbreviation of $PG(r-1,q)$. For a subset G of $E(P_r)$, let $R = E(P_r) - G$. We view the elements of G and R as being colored green and red, respectively. We call $P_r|G$ a *projective target* if there is a sequence $(F_0, F_1, \dots, F_r)$ of projective flats, that is, flats of P_r, with $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{r-1} \subseteq F_r$ and $r(F_j) = j$ for all j such that, for all i in $[r]$, the set $F_i - F_{i-1}$ is contained in either G or R. Such matroids were studied by Nelson and Nomoto [10] in the binary case and by Mizell and Oxley [8] for $GF(q)$-representable matroids when $q \geq 2$. Although $GF(q)$-representable matroids need not be uniquely $GF(q)$-representable, it was shown by Mizell and Oxley [8, Proposition 6] that if one $GF(q)$-representation of a simple $GF(q)$-representable matroid is a target, then all of the $GF(q)$-representations of M are targets. Let M be the projective target associated with the sequence $(F_0, F_1, \dots, F_r)$ of projective flats. Then, for any basis $\{x_1, x_2, \dots, x_r\}$ for P_r such that $\{x_1, x_2, \dots, x_i\}$ spans F_i for each i, all the elements of $F_{i+1} - F_i$ have the same color as x_{i+1}.

Given a simple $GF(q)$-representable matroid M of rank at most r, it is convenient to view M as a restriction of P_r. For this, we will take (G, R) to be a partition of P_r such that $P_r|G \cong M$. We can obtain a tipped or tipless coning of M from this 2-coloring (G, R) of P_r by first viewing this P_r as a hyperplane H of P_{r+1} and then taking a point p of $E(P_{r+1}) - H$. Next, for each point z of H, we color each point z' of $E(P_{r+1}) - (H \cup p)$ that is on the line between p and z so that the colors of z and z' agree. Finally. for a tipped coning, we color p green, and, for a tipless coning, we color p red.

The following result gives a new constructive characterization of the class of $GF(q)$-projective targets. Using this, we show in Corollary 3.9 that there is a unique binary projective target on n elements for all non-negative integers n.

Theorem 3.7. *The class of $GF(q)$-projective targets is exactly the class of matroids that can be obtained from the empty matroid by a sequence of q-conings and tipless q-conings.*

Proof. Suppose M is a projective target of rank at most r. As projective targets are uniquely $GF(q)$-representable, we lose no generality in considering a 2-coloring (G, R) of $E(P_r)$ such that $P_r|G \cong E(M)$. Let $\{x_1, x_2, \dots, x_r\}$ be a basis of P_r such that $\{x_1, x_2, \dots, x_i\}$ spans a projective flat X_i, and the color of x_{i+1} coincides with the color of each of the points in $X_{i+1} - X_i$ for each i in $[r-1]$. Every element z of M can be uniquely written as a linear combination of $x_1, x_2, \ldots, x_r$, and the color of z matches that of x_j where j is the highest index of a basis vector used in this linear combination. For each i in $[r]$, let $y_i = x_{r-i+1}$. If y_1 is green, then $P_r|\{y_1\}$ corresponds to a tipped coning of the empty matroid. If y_1 is red, then $P_r|\{y_1\}$ corresponds to a tipless coning of the empty matroid. For each $i \geq 2$, the color of each point y of the flat Y_i that is spanned by $\{y_1, y_2, \ldots, y_i\}$ coincides with the color of y_j where j is the lowest index of a member of $\{y_1, y_2, \ldots, y_i\}$ that is used in the linear combination yielding y. Thus, inductively, we see that each flat Y_i may be built by starting with a 2-coloring of the projective flat Y_{i-1} and adding y_i as a coloop and then adding all of the projective points between y_i and the points of Y_{i-1} such that each added point has the same color as its corresponding point in Y_{i-1}. We conclude that each projective target of rank at most r can be obtained from the empty matroid by a sequence or r operations each of which is a tipped or a tipless coning.

On the other hand, suppose N is a matroid that can be obtained from the empty matroid by a sequence of r operations, each a tipped or tipless coning. Let the tips, in order, be $y_1, y_2, \ldots, y_r$, where these are colored green or red if the coning is tipped or tipless, respectively. Every element z of N can be written as a linear combination of $y_1, y_2, \ldots, y_r$, and the color of z matches that of y_j where j is the lowest index of an element used in this linear combination. For each i in $[r]$, let $x_i = y_{r-i+1}$. Now let X_i be the projective flat that is spanned by $\{x_1, x_2, \ldots, x_i\}$. Then X_1, which equals $\{y_r\}$, is a tipped or tipless coning of the empty matroid depending on whether y_r is green or red. For each $i \geq 2$, the points in $X_i - X_{i-1}$ can all be written as linear combinations of $\{x_1, x_2, \ldots, x_i\}$. Moreover, each such linear combination must contain x_i. This implies that i is the highest index used in the linear combination. Since the highest indexed x_i used determines the lowest indexed y_{r-i+1} used, the color of each element in $X_i - X_{i-1}$ coincides with the color of x_i. We conclude that $P_r|G$ is a $GF(q)$ projective target. □

The empty projective target can be associated with the 0-1 string whose sole entry is 0. The following result is used in the proof of Corollary 3.9.

Corollary 3.8. *A non-empty projective target can be uniquely represented, up to isomorphism, by a* 0-1 *string whose leftmost entry is a* 1*.*

Proof. By Theorem 3.7, non-empty projective targets are exactly the matroids that can be obtained by starting with the empty matroid and repeatedly coning with or without a tip so that at least one coning has a tip. To construct the matroid corresponding to a particular 0-1 string that begins with a 1, we read the string from left to right interpreting each 1 as a tipped coning and each 0 as a tipless coning. □

Corollary 3.9. *For each non-negative integer n, there is, up to isomorphism, a unique binary projective target on n elements.*

Proof. As noted above, when $n = 0$, we associate the string 0 with the empty matroid. Now suppose $n \geq 1$. Then n has a binary expansion as a 0-1 string whose leftmost entry is a 1. By Corollary 3.8, there is a unique non-empty binary projective target with this 0-1 string. If $n = 1$, then the 0-1 string is 1, which corresponds to a tipped coning of the empty matroid, so the resulting matroid has exactly one element. Suppose Corollary 3.9 holds for 0-1 strings of length less than k that have 1 as their leftmost entry. Take a 0-1 string S of length k having 1 as its leftmost entry. Let N be the unique binary projective target corresponding to the 0-1 string S' obtained by deleting the rightmost entry of S. Then, by coning N, we get a matroid having $2|E(N)|$ elements if the coning is tipless, or having $2|E(N)| + 1$ elements if the coning is tipped. These two possibilities correspond to the two choices for S, which are obtained from S' by adjoining a 0 or a 1, respectively, as the rightmost entry. □

4. Avoiding $M(K_4)$ as an induced minor

In this section, we show that the class EXIM($M(K_4)$) is closed under taking generalized parallel connections across projective geometries, coning with a tip, and, for triangle-free matroids, coning without a tip. Unless otherwise stated, all matroids in this section are assumed to be binary. Let $\mathcal{N} = \text{EXIM}(M(K_4))$, the class of binary matroids that do not contain $M(K_4)$ as an induced minor. Recall that direct sum is a special case of generalized parallel connection.

Lemma 4.1. *The class $\mathcal{N}$ is closed under taking generalized parallel connections of members of $\mathcal{N}$ across projective geometries.*

Proof. Let M be a smallest-rank non-member of $\mathcal{N}$ such that there are matroids M_1 and M_2 in $\mathcal{N}$ and a projective geometry N with $M = P_N(M_1, M_2)$. Suppose, for some element e of $E(M)$, the matroid M/e has $M(K_4)$ as an induced minor. Then, by symmetry, we may assume that $e \in E(M_1) - E(N)$ or $e \in E(N)$. In the first case, $M/e = P_N(M_1/e, M_2)$, so M/e is a generalized parallel connection of two matroids in $\mathcal{N}$ across a projective geometry. Because M/e has smaller rank than M, we conclude that M/e does not have $M(K_4)$ as an induced minor, a contradiction. Now suppose that $e \in E(N)$. Then $M/e = P_{N/e}(M_1/e, M_2/e)$. Since N/e is a projective geometry and both M_1/e and M_2/e are in $\mathcal{N}$, we deduce that $M/e \in \mathcal{N}$, a contradiction. Hence $M/e \in \mathcal{N}$ for all e in $E(M)$. As $M \notin \mathcal{N}$, we see that M must have a proper flat F such that $M|F$ is isomorphic to $M(K_4)$. Now $F \cap E(M_1)$ is a flat of M_1 and $F \cap E(M_2)$ is a flat of M_2. Without loss of generality, we may assume that $E(M_1)$ contains a basis of $M|F$. Hence, F is contained in $E(M_1)$. Therefore, M_1 has $M(K_4)$ as an induced minor, a contradiction as $M_1 \in \mathcal{N}$. $\square$

The next two results identify some structure of matroids that have a coning point, that is, a point f such that f is in a triangle with every other element of the matroid.

Lemma 4.2. *If N has an element x in a triangle with every element and N has a flat F such that $N|F$ is isomorphic to $M(K_4)$, then $N|\mathrm{cl}_N(F \cup \{x\})$ is isomorphic to a rank-4 projective geometry with two points deleted.*

Proof. The element x cannot be in F since every element of F is contained in a rank-2 flat of size 2 but every rank-2 flat of N that contains x is a triangle. Thus, $r(\mathrm{cl}_N(F \cup \{x\})) = 4$ and the result follows since every element of F is in a triangle with x. $\square$

The next lemma shows that the class $\mathcal{N}$ is closed under an operation that is very similar to coning.

Lemma 4.3. *Let M be a rank-r matroid in $\mathcal{N}$ and suppose that, for some x in $E(M)$, the matroid N is obtained by adding every element on the line $\mathrm{cl}_{P_r}(\{x, y\})$ for each y in $E(M) - x$. Then N is in $\mathcal{N}$.*

Proof. Suppose that, for some subset X of $E(N)$, the matroid N/X has a flat F such that $(N/X)|F \cong M(K_4)$. Since M/x does not have $M(K_4)$ as an induced minor, and $N/x \cong M/x$, we may assume $x \in E(N/X)$. Then, by Lemma 4.2, N/X has a flat F' containing x such that $(N/X)|F'$ is isomorphic to a rank-4 projective geometry with two points deleted. Then, by contracting x, we obtain $M(K_4)$ as an induced minor of M, a contradiction. $\square$

Recall that, when N is a simple binary matroid, $A(N)$ is the coning of N with tip p, and $A(N)\backslash p$ is the tipless coning of N.

Lemma 4.4. *If N is in $\mathcal{N}$, then $A(N)$ is in $\mathcal{N}$.*

Proof. Let N be a smallest-rank member of $\mathcal{N}$ such that $A(N)$ is not in $\mathcal{N}$. Suppose, for some e in $E(A(N))$, the matroid $A(N)/e$ has $M(K_4)$ as an induced minor. As $A(N)/p \cong N$, we see that $e \neq p$. Then, by Lemma 3.1(i), e is in a hyperplane H of $A(N)$ such that $A(N)|H \cong N$. Hence we may assume that $e \in E(N)$. Then $A(N)/e \cong A(N/e)$. By the choice of N, the latter is in $\mathcal{N}$. Hence so is the former, a contradiction.

We conclude that $A(N)$ contains a flat F such that $A(N)|F \cong M(K_4)$. As $A(N)$ has an element, namely p, that is in a triangle with every other element of $E(A(N))$, it follows, by Lemma 4.2, that $A(N)$ has a rank-4 flat F' containing p that is isomorphic to a rank-4 projective geometry with two points deleted. Hence $(A(N)|F')/p$ is isomorphic to $M(K_4)$. Since p was contracted to produce this induced minor, we conclude that $(A(N)|F')/p$ is isomorphic to an induced minor of N, a contradiction. □

Lemma 4.5. *If N is a triangle-free member of $\mathcal{N}$, then $A(N)\backslash p \in \mathcal{N}$.*

Proof. Suppose N is a smallest-rank triangle-free member of $\mathcal{N}$ such that $A(N)\backslash p$ is not in $\mathcal{N}$. By Lemma 3.1(i), each element e of $E(A(N))$ is contained in a hyperplane H isomorphic to N. Hence $(A(N)\backslash p)/e$ is isomorphic to $A(N/e)$, and, by Lemma 4.4, $A(N/e)$ is in $\mathcal{N}$. We deduce that $A(N)\backslash p$ has $M(K_4)$ as an induced restriction. However, as N has no triangles, $A(N)\backslash p$ has no triangles. Hence $A(N)\backslash p$ does not have $M(K_4)$ as an induced restriction. □

On combining Lemmas 4.1, 4.4, and 4.5, we immediately obtain the following result.

Corollary 4.6. *The class $\mathcal{N}$ of binary matroids that do not have $M(K_4)$ as an induced minor is closed under the following operations.*

(i) Generalized parallel connections across projective geometries;
(ii) tipped coning; and
(iii) tipless coning of triangle-free matroids.

The next lemma [12, Corollary 3.7] will be used in the proofs of the two subsequent lemmas.

Lemma 4.7. *Let M be a 3-connected binary matroid having rank and corank at least three and suppose that $\{x, y, z\} \subseteq E(M)$. Then M has a minor isomorphic to $M(K_4)$ whose ground set contains $\{x, y, z\}$.*

Lemma 4.8. *No 3-connected member of $\mathcal{N}$ has a triad.*

Proof. Let T^* be a triad of a member M of $\mathcal{N}$. Then, by Lemma 4.7, T^* is in an $M(K_4)$-minor of M. As $(T^*, E(M) - T^*)$ is a 3-separation of

M, such a minor is obtained by contracting elements from $E(M)-T^*$, so this $M(K_4)$-minor is an induced minor of M, a contradiction. $\square$

Recall that, for a rank-r binary matroid M that is viewed as a restriction of P_r, if $X \subseteq E(P_r) - E(M)$, we denote by $M + X$ the matroid $P_r|(E(M) \cup X)$. For the rest of the paper, when we write M/e, we mean the matroid obtained from M by contracting the element e but without simplifying the result. The simplification of M/e is denoted by $\mathrm{si}(M/e)$.

Lemma 4.9. *Let (X, Y) be a vertical 3-separation in a 3-connected matroid M. Let $G_P = \mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y)$ and $G_M = \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$. If $|G_M| \geq 1$, then M has both $(M|\mathrm{cl}_M(X)) + (G_P - G_M)$ and $(M|\mathrm{cl}_M(Y)) + (G_P - G_M)$ as induced minors.*

Proof. Let $G_P = \{a, b, c\}$. Then $G_M \subseteq G_P$ and we may assume that $a \in G_M$. Let $M_X = (M|\mathrm{cl}_M(X)) + (G_P - G_M)$ and $M_Y = (M|\mathrm{cl}_M(Y)) + (G_P - G_M)$. Then it is well known and straightforward to check that each of M_X and M_Y are 3-connected. Thus, by Lemma 4.7, $\{a, b, c\}$ is contained in an $M(K_4)$-minor of M_X. Hence, M_X has an induced minor N that contains $\{a, b, c\}$ and is isomorphic to $M(K_4)$ or F_7. Then N has an element x such that N/x has $\{a, b, c\}$ as a triangle and has elements in parallel with each of b and c. By performing each of these operations in $M + (G_P - G_M)$ and subsequently deleting $G_P - G_M$, it follows that M_Y is an induced minor of M, and the lemma follows by symmetry. $\square$

For $r \geq 3$, a *binary r-spike with tip t* is the vector matroid M_r of the binary matrix $[I_r|J_r - I_r|\mathbf{1}]$ where J_r and $\mathbf{1}$ are the $r \times r$ and $r \times 1$ matrices of all ones, respectively, and $\mathbf{1}$ is labeled by t. Note that M_r/t is obtained from an r-element circuit by replacing every element by two elements in parallel. It is not difficult to check that all matroids of the form $M_r \backslash z$, where $z \in E(M_r) - \{t\}$, are isomorphic. We call $M_r \backslash z$ a *binary r-spike with tip t and cotip t^**, where $\{t, z, t^*\}$ is a triangle of M_r. Clearly $M_3 \backslash z \cong M(K_4)$. Also, one can show that $M_r \backslash z$ is self-dual for all r.

Lemma 4.10. *Let M be a simple matroid having an element e such that M/e has a basis B each element of which is in a 2-circuit. If M/e has at least one element that is not in a 2-circuit, then M has an induced minor isomorphic to a spike with a tip and cotip.*

Proof. Let f be an element of M/e that is not in a 2-circuit. Then, in M/e, the fundamental circuit of f with respect to B has every element except f in a 2-circuit. Let C be a smallest circuit of M/e for which, with exactly one exception, every element is in a 2-circuit. Let g be the exceptional element of C. Then the minimality of C implies that $M|\mathrm{cl}_M(C \cup \{e\})$ is a spike with tip e and cotip g. $\square$

Corollary 4.11. *Let M be a rank-r matroid with $r \geq 3$. If M has an element e such that M/e has a basis, each element of which is in a 2-circuit, and M/e has at least one element that is not in a 2-circuit, then M has $M(K_4)$ as an induced minor.*

The rest of this section is devoted to proving that if a 3-connected matroid M has an element e such that neither M nor $M\backslash e$ has $M(K_4)$ as an induced minor, then every element of M is on a triangle through e. In this case, since every element of $M\backslash e$ is on a triangle through e, and M does not have $M(K_4)$ as a flat, we deduce that M is a coning with tip e of a triangle-free matroid.

Proof of Theorem 1.7. We argue by induction of $r(M)$. The result is vacuously true when $r(M) = 3$. Assume the result holds for $r(M) \leq r-1$ and let M be a matroid satisfying the hypothesis and having $r(M) = r$. We may assume there is an element f such that $\mathrm{cl}_M(\{e, f\}) = \{e, f\}$. Let t denote the point of $P_r - E(M)$ on the line spanned by $\{e, f\}$.

4.12.1. $\mathrm{si}(M/f)$ *is 3-connected.*

Suppose $\mathrm{si}(M/f)$ is not 3-connected. Then M has a vertical 3-separation (X, Y) with $f \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Let $G_P = \mathrm{cl}_{P_r}(X) \cap \mathrm{cl}_{P_r}(Y)$. Suppose first that $e \notin G_P$. Then, without loss of generality, $e \in X$. By Lemma 4.9, M has $(M|\mathrm{cl}_M(X)) + (G_P - E(M))$ as an induced minor. By the choice of M, every element of $(M|\mathrm{cl}_M(X)) + (G_P - E(M))$ is in a triangle with e. In particular, every element of the triangle G_P is in a triangle with e. Therefore, $M\backslash e$ has $M(K_4)$ as an induced minor, a contradiction. Hence, we may assume that $e \in G_P$. Let $M_1 = M|\mathrm{cl}(X)$ and $M_2 = M|\mathrm{cl}(Y)$. By Lemma 4.9, $M_1 + t$ is an induced minor M_1' of M. Since M_1' is 3-connected and neither M_1' nor $M_1'\backslash t$ has $M(K_4)$ as an induced minor, the choice of M implies that every element of M_1 lies in a triangle of $M_1 + t$ with t. By symmetry, every element of M_2 lies in a triangle of $M_2 + t$ with t.

We show next that every element of $E(M_2) - \{e, f\}$ is in a triangle of M_2 with e. Assume that an element y of $E(M_2) - \{e, f\}$ is not in such a triangle. Then M/y is M_1' with no element parallel to e, and M_1' is a proper induced minor of M. We deduce that neither M_1' nor $M_1'\backslash e$ has $M(K_4)$ as an induced minor. Hence every element of $M_1 + t$ is in a triangle with e. Let a be an element of $E(M_1) - \{e, f\}$. Then M_1 has elements b and c such that $\{a, b, e\}$ and $\{a, c, t\}$ are triangles of

M_1+t. Thus, there is an additional element d of M_1 that is on the line $\mathrm{cl}(\{c,e\})$. Hence $\mathrm{cl}_M(\{e,f,a\})$ is a rank-3 flat containing six points, a contradiction. We conclude that every element of $E(M_2)-\{e,f\}$ is in a triangle with e. Thus, M_2/e has a basis, each element of which is in a 2-circuit, and f is not in a 2-circuit of M_2/e. By Corollary 4.11, M_2 has $M(K_4)$ as an induced minor. Hence 4.12.1 holds.

Let $M_f=\mathrm{si}(M/f)$. Then, by 4.12.1, M_f is a 3-connected induced minor of M, and neither M_f nor $M_f\backslash e$ has $M(K_4)$ as an induced minor. By the choice of M, every element of M_f is in a triangle with e. For a triangle $\{x,y,e\}$ of M_f, either $\{x,y,e\}$ or $\{x,y,e,f\}$ is a circuit of M. From this, we deduce the following.

4.12.2. *Every element of M is in a triangle of $M+t$ with at least one of e and t.*

Next we prove the following assertion.

4.12.3. *In $M+t$, among the triangles other than $\{e,f,t\}$, no triangle containing e meets a triangle containing t.*

To see this, first suppose there are elements a,b, and c in $E(M)$ such that $\{a,b,e\}$ and $\{a,c,t\}$ are distinct triangles in $M+t$. Then, as M has no $M(K_4)$ induced minor, there is an element s of $P_r-E(M+t)$ in the rank-3 projective flat F' spanned by $\{e,f,a\}$.

Assume there is a triangle $\{x,y,t\}$ in $M+t$ that is not contained in F'. As neither $\mathrm{si}(M/x)$ nor $\mathrm{si}(M/y)$ has $M(K_4)$ as a flat, M has elements z and w such that $\{x,w,s\}$ and $\{y,z,s\}$ are triangles of $M+s$. The projective flat F spanned by $\{e,f,a,b,c,x,y,z,w\}$ has rank 4. Its restriction to the nine listed points is isomorphic to the unique single-element extension of $AG(3,2)$.

We show next that $E(M)\cap F=\{a,b,c,e,f,w,x,y,z\}$. Assume this fails. Then, as neither s nor t is in $E(M)$, the matroid $M|(F\cap E(M))$ must be isomorphic to the complement in P_4 of one of $U_{2,2}$, $U_{2,3}$, $U_{3,3}$, $M(C_4)$, $U_{1,1}\oplus U_{2,3}$, or the single-element deletion of $M(K_4)$. In each case except when the complement is $M(C_4)$, the matroid $M|(F\cap E(M))$ has a flat isomorphic to $M(K_4)$, a contradiction. In the exceptional case, $(M\backslash e)|(F\cap E(M\backslash e))$ is isomorphic to the complement in P_4 of $M(C_4)\oplus U_{1,1}$ or of the single-element deletion of $M(K_4)$. In these two cases, $(M\backslash e)|(F\cap E(M\backslash e))$ has a flat isomorphic to $M(K_4)$, a contradiction. We conclude that $E(M)\cap F=\{a,b,c,e,f,w,x,y,z\}$.

Let $M'=M|\{a,b,c,e,f,w,x,y,z\}$. Then M'/x is an induced minor of M. This matroid is obtained from a copy of F_7 with ground set $\{a,b,c,e,f,w,x,y\}$ by adding z in parallel to b. In particular, in M'/x, the element e is in a parallel class of size one, so $\mathrm{si}(M'/x\backslash e)$ is an induced minor of $M\backslash e$ that is isomorphic to $M(K_4)$, a contradiction. We deduce that every triangle of $M+t$ containing t is contained in F'.

We now know that every element of $E(M)-\{c,f\}$ is in a triangle of M containing e. As M is 3-connected, it has $\{c,f\}$ as a coindependent set. Hence M/e has a basis that avoids $\{c,f\}$. Therefore, M/e has a basis each element of which is in a 2-circuit. As c is an element of M/e that is not in a 2-circuit, Corollary 4.11 implies that M has $M(K_4)$ as an induced minor, a contradiction. We conclude that 4.12.3 holds.

4.12.4. *M has no triangle containing f.*

Assume that M has such a triangle T. Then one easily checks using 4.12.2 that, in $M+t$, the flat spanned by $T\cup\{e\}$ contains distinct intersecting triangles other than $\{e,f,t\}$, one containing e and one containing t, a contradiction to 4.12.3.

Let $\{g,h,t\}$ be a triangle of $M+t$ different from $\{e,f,t\}$. Then, by 4.12.3 and 4.12.4, both $\mathrm{cl}_M(\{e,g\})$ and $\mathrm{cl}_M(\{f,g\})$ contain exactly two elements. Let u and s be the points in $P_r-E(M)$ on the lines $\mathrm{cl}_{P_r}(\{e,g\})$ and $\mathrm{cl}_{P_r}(\{f,g\})$, respectively. Then, by replacing $\{e,f,t\}$ by $\{e,g,u\}$ and by $\{e,h,s\}$, we deduce from 4.12.2 and 4.12.3 that, in $M+u$, every element of $E(M)-g$ is in a triangle with exactly one of e and u; and, in $M+s$, every element of $E(M)-h$ is in a triangle with exactly one of e and s.

4.12.5. *$M+t$ has at most one triangle other than $\{e,f,t\}$ containing t.*

Suppose that $\{x,y,t\}$ is a triangle of $M+t$ that differs from both $\{e,f,t\}$ and $\{g,h,t\}$. Since M does not have $M(K_4)$ as an induced restriction, $r(\{x,y,e,f,g,h\})=4$. By 4.12.3, for the triangles $\{e,x,i\}$ and $\{e,y,j\}$ of P_r, neither i nor j is in $E(M)$. Thus, as neither x nor y is on a triangle through e, it follows by the preceding paragraph that $M+u$ has triangles $\{u,x,z\}$ and $\{u,y,w\}$ for some z and w in $E(M)$. Now $M|\{e,f,g,h,x,y,w,z\}\cong AG(3,2)$. Let $F=\mathrm{cl}_M(\{e,f,g,h,x,y,w,z\})$. Then $\mathrm{cl}_{P_r}(F)-F$ contains $\{i,j,s,t,u\}$ and $P_r|\{i,j,s,t,u\}$ is a single-element deletion of $M(K_4)$. It follows that $M|F$ is one of $AG(3,2)$, the unique rank-4 single-element extension of $AG(3,2)$, or the unique rank-4 two-element extension of $AG(3,2)$. In the last case, the matroid $M+t$ has a triangle containing e that meets a triangle containing t and does not contain $\{e,t\}$, a contradiction to 4.12.3.

In the second case, $M|F$ is a rank-4 binary spike with a tip and this tip is not equal to e. In that and the first case taking a to be an element of F that differs from e and from the spike tip when it is present, we have that $\mathrm{si}((M|F)/a\backslash e) \cong M(K_4)$, a contradiction. Hence 4.12.5 holds.

We now know that either every element of M except f is in a triangle with e, or M has exactly three elements f, g, and h that are not in triangles with e. In the first case, M/e certainly has a basis each element of which is in a 2-circuit. As $\{e, f\}$ is a flat of M, by Corollary 4.11, M has an induced minor isomorphic to $M(K_4)$, a contradiction. We deduce that f, g, and h are the only elements of $E(M) - e$ that are not in a triangle with e. By Lemma 4.8, M has no triads. Hence $r(M) = r(M\backslash\{f, g, h\})$. Thus, M/e has a basis avoiding $\{f, g, h\}$. Each element of this basis is in a 2-circuit of M/e. Also, f is in a parallel class of size one in M/e. Thus, by Corollary 4.11, M has $M(K_4)$ as an induced minor, a contradiction. $\square$

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